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DISCRETE FOURIER TRANSFORM AND ITS PROPERTIES- A REVIEW

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Abstract: **In this paper attention is paid to Fourier and Discrete Fourier Transform. Definition, properties, applications are described. Relation between Z Transform with DFT is given.**

Key words: **Fourier Transform, Discrete Fourier Transform**

1. INTRODUCTION

The Fourier Transform is most widely used tools for transferring data sequences. It is used for decomposing signals into its constituent frequencies and its oscillatory functions. It represent signal in frequency domain and transform one complex value function of real variable into another Fourier transform.

The goal of the paper is to give basic information of Discrete Fourier Transform. We explain properties and examples of DFT. Application of designing filter for noise reduction in audio signals.

2. DEFINITION - FOURIER SERIES

The representation of periodic function (function defined only on finite interval) as linear combination of sine and cosines is known as Fourier series expansion of the function. Generally defined as

$$\mathrm{f(t)} = \sum_{1}^{N}(\mathrm{A_n}\cos(2\pi \mathrm{w_n t}) + B_n \sin(2\pi \mathrm{w_n t})) \quad \textbf{(1)}$$

Sine function in the form $B_n \sin(2\pi wt + \varphi)$ has B as amplitude; w frequency measured in cycles per second and φ is an angle or phase used to get value other than zero. Cosine function has same component as sine with angle $\frac{\pi}{2}$.

A. FOURIER SERIES IN COMPLEX FORM

Above equation (1) can be written as

$$\mathrm{f(t)} = \sum_{n=1}^{N} \left(\frac{\mathrm{A_n}}{2} \left(e^{i2\pi \mathrm{w_n t}} + e^{-i2\pi \mathrm{w_n t}} \right) + \frac{B_n}{2i} \left(e^{i2\pi \mathrm{w_n t}} - e^{-i\,2\pi \mathrm{w_n t}} \right) \right. \quad \textbf{(2)}$$

Let $c_n = \frac{A_n + iB_n}{2}$, for $\mathrm{n} > 0$, $c_n = \frac{A_n - iB_n}{2}$ for $\mathrm{n} < 0$, $c_0 = 0$, $w_n = -w_{-n}$ for $\mathrm{n} < 0$ f(t) takes the form

$$\mathrm{f\,(t)} = \sum_{n=-k}^{k} c_n \; e^{i2\pi \mathrm{w_n t}} \quad \textbf{(3)}$$

3. CONTINUOUS FOURIER TRANSFORM

Fourier transform of f(t) defined as

$$\mathrm{F(w)} = \int_{-\infty}^{\infty} \mathrm{f}(t)\, e^{-2\pi iwt}\, \mathrm{dt} \quad \textbf{(4)}$$

Inverse Fourier transform of F(w) is given by

$$\mathbf{f(t)} = \int_{-\infty}^{\infty} \boldsymbol{F(w)}\, \boldsymbol{e^{2\pi iwt}}\, \mathbf{dw} \quad \textbf{(5)}$$

The result applying Fourier transform to the function is called frequency spectrum or power spectrum of the function.

Generally, w = $2\pi v$ (time angular frequency) $\hat{F}$ (w) =F (v) = F ($\frac{w}{2\pi}$) = $\int_{-\infty}^{\infty} f(t)\, e^{-iwt}$ dt , where
f(t) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(w)\, e^{iwt}\, d(w)$ **(6)**

3.1 SOME FUNCTION AND ITS FOURIER TRANSFORM

	Function	Fourier Transform F(w)
1	$\delta(x) = 1 \quad for\ x \geq 0$ $= 0 \quad for\ x$	1
2	cos ($2\pi kx$)	$\frac{1}{2}$ $[\delta(w-k) + \delta(w+k)]$
3	sin ($2\pi kx$)	$\frac{1}{2i}$ $[\delta(w-k) - \delta(w+k)]$
4	f (x) = A $\quad -\frac{x}{2} \leq t \leq \frac{x}{2}$ = 0 otherwise	Ax [$\frac{\sin(\frac{wx}{2})}{(\frac{wx}{2})}$]
5	$f(x) = e^{-ax}$, $t > 0$	$\frac{1}{a+iw}$

4. PROPERTIES OF FOURIER TRANSFORM

i) Linearity property: If Fourier transform of f (x) is F (w)**,**The F[a $f_1(x)$ + b f_2 (x)] = a $F_1(w)$+bF_2 (w), where a & b are constant.

ii) Convolution Theorem: Convolution theorem states that the Fourier transform of the convolution of two functions is equal to the product of the Fourier transforms of the original functions. F **(**$f_1(x) * f_2(x)$) = $F_1(w)$ $F_2(w)$.

Note – convolution of function f(x) and g(x) is given by f(x) $*$ g(x) = $\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$

iii) Product theorem: The Product theorem states that the Fourier transform of product of two functions is equal to the convolution of Fourier transforms of the original functions.ie F ($f_1(x)$ $f_2(x)$) = $F_1(w) * F_2(w)$. Convolution Theorem & Product theorem states that convolution in time domain corresponds to a multiplication of coefficient in frequency domain and vice versa.

iv) Time shifting Property: Fourier transform of f (x- k) is F (w). $e^{-2\pi iwk}$. It states that total energy of signal is same as original function in time corresponds to a change in phase of sinusoids comprising functions.

The proofs of properties are given by many researchers (Wea 83, OS 8, Jac 90). Generally physical signals are continuous and have continuous range of frequencies. The functions that are used to model them are continuous and having continuous transform. In many cases discrete values at various points in time are relatively easy to obtain and computers which process such data are inherently discrete as well. We need transform from the discrete time domain to the discrete frequency domain and an inversely. Therefore we discuss the discrete signals and their transforms.

5. DISCRETE FOURIER TRANSFORM (DFT)

DFT is specific kind of discrete transform used in Fourier analysis. It Transforms one function into another, which is called frequency domain representation of original function which is often a function in time domain. DFT requires input function which is discrete. DFT is widely employed in signal processing and related field to analyze frequency contained in a sampled signals.

5.1 DEFINITION

Let X (e^{jw}) be discrete time Fourier transform of discrete time signal x (n). DFT of x(n) obtained by sampling one period of discrete time Fourier transform X (e^{jw}) at finite number of frequency points. Let x (n) = discrete time signal of length L. X (k) = DFT of x (n). Let N point DFT of x(n) , N $\geq L$ defined as DFT {x (n)} = X (k) = $\sum_{n=0}^{N-1} x(n)\ e^{\frac{-j2\pi kn}{N}}$, for k=0, 1, 2, 3…N-1. where X (k) is sequence consist N complex number for k = 0, 1, 2, 3… ..N-1, X(k) = { X(0), X(1), X(2),………………….. X (N-1)}.

5.2 INVERSE DFT

The Inverse DFT of X(k) of length N is given by x(n) = $\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$, for n = 0,1,2,3,… N-1.

5.3 FREQUENCY SPECTRUM

X (k) is discrete function of frequency of discrete time signal w. It is called Frequency spectrum or signal spectrum of x(n). Let X(k) is complex valued function of k , X (k) = $X_r(k) + X_j(k)$ i.e. real part + imaginary part. Where Magnitude spectrum is $|X(k)|^2$=X(k)X*(k)=$\sqrt{x_r^2(k) + x_j^2(k)}$, Where X*(k) is complex conjugate of X (k). Phase spectrum ∠ X(k) is argument of $X(k) = \tan^{-1}\left[\frac{x_j(k)}{x_r(k)}\right]$

∴ Magnitude sequence |X(k)| = { |X(0)|, |X(1)|, |X(2)|,…… |X(N-1)|}

∴ phase Sequence ∠ X (k) = {∠ X (0), ∠ X (1), ∠ X(2)……. ∠ X(N-1)}

Plot of samples of magnitude sequence verses k is called magnitude Spectrum and plot of Phase sequence verses k is called Phase spectrum.

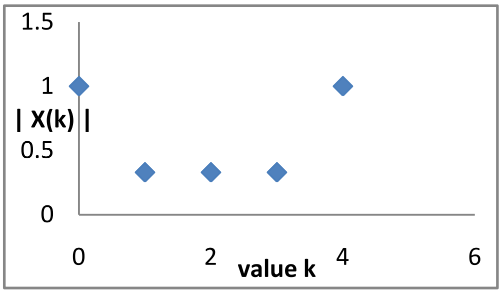
Example 1) consider 4 point DFT for sequence,

x (n) = $\frac{1}{3}$ for $0 \leq n \leq 2$

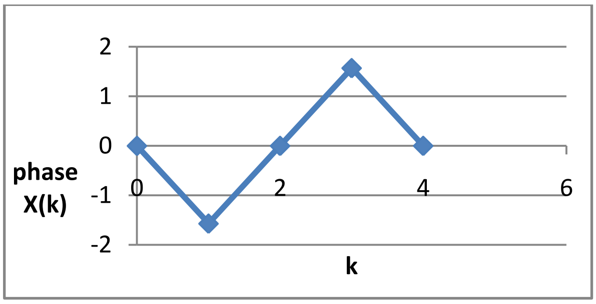
= 0 otherwise

Magnitude Sequence |X (k)| = {1,0.33,0.33,0.33,1}
Phase Sequence ∠X(k) ={0,-1.571 , 0, 1.571}

Magnitude spectrum



Phase spectrum

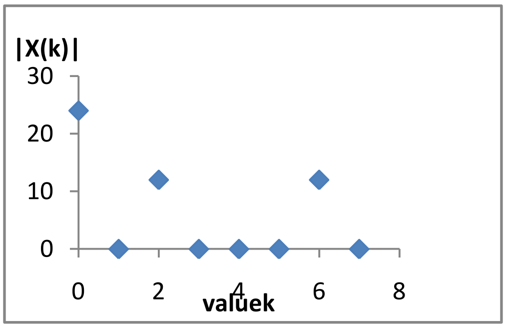


Example 2 Consider x (n) = 6 $cos^2\left(\frac{\pi n}{4}\right)$

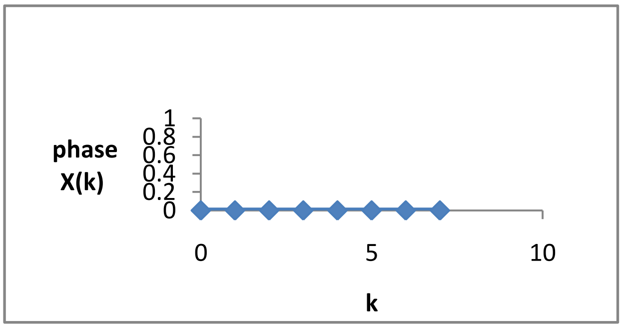
Magnitude sequence | X (k)| = {24, 0, 12, 0, 0, 0, 24, 0}

Phase sequence ∠ X (k) = { 0,0,0,0,0,0,0}

Magnitude spectrum



Phase spectrum



Example 3

Consider sequence x(n) = {4, 3, 2, 1 }

x(n) = 4- n $0 \leq n \leq 3$

= 0 otherwise

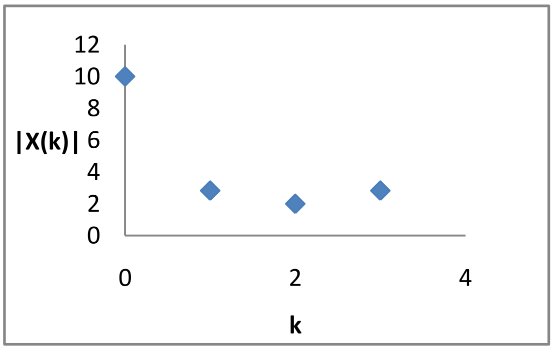
Magnitude sequence |X(k)| = { |X(0)|, |X(1)|, |X(2)|, |X(3)|}

∴X (k) = { 10, 2.828, 2 , 2.828 }

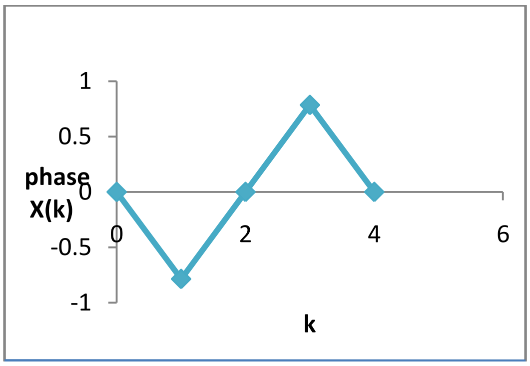
∴ phase Sequence ∠ X (k) = { ∠ X(0), ∠ X(1), ∠ X(2), ∠ X(3)}

∴ ∠ X (k) = { 0,-0.785 , 0, 0.785 }

Magnitude Spectrum



Phase spectrum



6. .DFT AS LINEAR TRANSFORMATION

Consider $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$ for k=0, 1, 2, 3...N-1

Let $w \in (0,2\pi)$ and $W_N = e^{\frac{-j2\pi}{N}}$ N^{th} root of unity

$\therefore X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$ for k=0,1,2,3,……..N-1

X and x are (N x 1) matrices, and W_N is an (N x N) square matrix called the DFT matrix. Generally matrix form is given by

$X(k) = W_N\, x(n)$

Where $X(k) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \dots \\ \dots \\ X(N-1) \end{bmatrix}$ $x(n) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \dots \\ \dots \\ x(N-1) \end{bmatrix}$ $W_N = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \dots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \dots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ W_N^0 & W_N^0 & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix}$ X(k) = W_N

x(n) written as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \dots \\ \dots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^1 & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 1 & W_N^0 & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \dots \\ \dots \\ x(N-1) \end{bmatrix}$$

Example 1 To find 4 point DFT for x(n) = (1,2,1,0)

For N=4 $W_4 = e^{\frac{-j2\pi}{4}} = e^{-\frac{j\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 0 - j\,1 = -j$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & 1 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Now $W_4^1 = (-j)^1 = -j$, $W_4^2 = (-j)^2 = -1$, $W_4^3 = (-j)^3 = j$, $W_4^4 = (-j)^4 = 1$

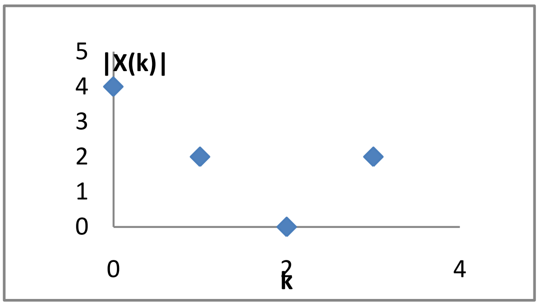
$\therefore X(k) = \{4, -2j, 0, 2j\}$

Magnitude sequence $|X(k)| = \{|X(0)|, |X(1)|, |X(2)|, |X(3)|\}$

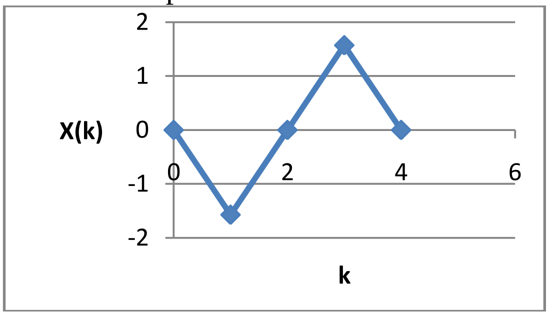
$\therefore X(k) = \{4, 2, 0, 2\}$ $\therefore$ Phase Sequence $\angle X(k) = \{\angle X(0), \angle X(1), \angle X(2), \angle X(3)\}$

Phase Sequence $\angle X(k) = \{0, -0.5\pi, 0, 0.5\pi\} = \{0, -1.571, 0, 1.571\}$

Magnitude Spectrum



Phase spectrum



7. PROPERTIES OF DFT

Applying the DFT to combination of two periodic sequences $x_1(n)$, $x_2(n)$ we have check periodicity of the combination. Since DFT is defined over single period ,for the DFT of combination to be well defined, it must have single periodicity. There are three forms of combination in continuous case, linear combination $ax_1+bx_{2,}$ Convolution of x_1 & $x_{2,}$ multiplication x_1 $x_{2.}$ Both linear combination and multiplication for continuous case are defined as $x_1(n)$ is paired with $x_2(n)$.similarly in discrete case each $x_1(i)$ should be combined with $x_2(i)$.Thus $x_1(n)$ and $x_2(n)$ are of same periodicity N. and resulting sequence is of periodicity N If two sequences have different periodicity N_1 & N_2 ,then using padding the sequence of periodicity N_1 is made of periodicity N_2 by adding zero's at end of N_1.Therefore combination is well defined.

i) Linearity Property Let $X_1(k)$ = DFT of $x_1(n)$ & $X_2(k)$ = DFT of $x_2(n)$

∴ DFT {a $x_1(n)$ + b $x_2(n)$ } = a $X_1(k)$ + b $X_2(k)$ a,b are constants

ii) Periodicity If a sequence x(n) periodic with periodicity N then N point DFT, X(k) is also periodic with periodicity N .Let x(n+N) = x(n) $\forall\ n$. Then DFT (X (k+N)) = X(k) $\forall\ k$

iii) Circular Time shift It states that if discrete time signal is circularly shifted in time by m units then it's DFT is multiplied by $e^{\frac{-j2\pi km}{N}}$ ie. If DFT x(n) = X(k) Then DFT {(x(n-m) mod N}=X(k) $e^{\frac{-j2\pi km}{N}}$

iv) Circular Frequency shift If discrete time signal multiplied by $e^{\frac{j2\pi nm}{N}}$ then DFT is circularly shifted by m units.ie. If DFT x(n) = X(k) Then DFT {x(n) $e^{\frac{j2\pi nm}{N}}$} = $X((k-m))_N$

v) Multiplication DFT of product of two discrete time sequence equivalent to circular convolution of DFT of individual sequences scaled by factor $\frac{1}{N}$.

ie If DFT x(n) = X (k), Then DFT {$x_1(n)$ x_2 (n)}= $\frac{1}{N}$ $\{X_1(k) \circledast X_2(k)\}$

8. RELATION BETWEEN DFT & Z TRANSFORM

The Z transform of N point sequence x(n) is Z{x(n)}= X(z)= $\sum_{n=0}^{N-1} x(n)z^{-n}$

Let X(z) at N equally spaced point on unit circle i.e. at z = $e^{\frac{j2\pi k}{N}}$.As $|e^{\frac{j2\pi k}{N}}| = 1$ and angle $of\ e^{\frac{j2\pi k}{N}} = \frac{2\pi k}{N}$.

Term z = $e^{\frac{j2\pi k}{N}}$ for k= 0, 1, 2…N-1 represents N equally spaced point on unit circle z = $e^{\frac{j2\pi k}{N}}$ in z plane.

$\therefore [X(z)]_{at\ z=e^{\frac{j2\pi k}{N}}} = [\sum_{n=0}^{N-1} x(n)z^{-n}]_{at\ z=e^{\frac{j2\pi k}{N}}}$

$= \sum_{n=0}^{N-1}$ x(n) $e^{\frac{-j2\pi kn}{N}}$

$\therefore [X(z)]_{at\ z=e^{\frac{j2\pi k}{N}}}$ = X (k)

∴ N point DFT of finite duration sequence can be obtained from Z transform of sequence by evaluating Z transform of the sequence at N equally spaced points around unit circle in z plane.

9. APPLICATION OF DFT

i) Filtering- To filter out noise from the observation vectors, before further processing them, a transformation is applied to each vector, which eliminates the high frequency components of the data.(usually "noise" corresponds to high frequency data while actual data is characterized by low frequency)The transformation is equivalent to applying DFT multiplying resulting vector by sequence such that lower frequency coefficient are multiplied by number close to zero. The inverse DFT applied to result we get single results a new vector transformation which is low pass filter. Multiplying each vector by this matrix results new vector which is same as original in its lower frequency but missing higher frequencies that is less noisy vector.

DFT provides discrete frequency representation of finite sequence in frequency domain so it use as computation tool for linear system analysis and for linear filtering.

ii) Coding Application- In coding application the DFT used in two broad classes, in power spectrum estimator and in sub band coding where it is used in implementation of complex cosine or sine modulated filter bands.

10. CONCLUDING REMARK

The Fourier Transform is a broad subject of which we have covered only a small fraction. We presenting the transform from a mathematical point of view and providing intuition through examples and applications. Fourier transform is a way to represent functions and sequences as a combination of sinusoids.

In mathematics, the discrete Fourier transform (DFT) converts a finite list of equally spaced samples of a function into the list of coefficients of a finite combination of complex sinusoids, ordered by their frequencies, that has those same sample values. It can be said to convert the sampled function from its original domain (often time or position along a line) to the frequency domain The DFT is the most important discrete transform, used to perform Fourier analysis in many practical applications. The DFT is also used to efficiently solve partial differential equations, and to perform other operations such as convolutions or multiplying large integers. Since it deals with a finite amount of data, it can be implemented in computers by numerical algorithms.

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