ELLIPSE PERIMETER: A NEW APPROXIMATE FORMULA BY A NEW APPROACH

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Abstract

In this paper I introduce a new formula for the perimeter of the ellipse. It is more precise than all known formulae of its category. My approach is through the well-known first order Partial Differential Equation, namely, Lagrange Equation. It is hoped that this approach will open up further research on the subject and would eventually end up with a handy exact formula.

Key words

Ellipse, Major and Minor Radii, Aspect ratio, Relative Error, Holder Norm, Holder mean.

I. TERMINOLOGY AND NOTATIONS

In this paper

- 'Ellipse' means the standard ellipse whose Cartesian equation is: $(x/a)^2 + (y/b)^2 = 1$, $a \geq b \geq 0$. 'a' and 'b' are the major and minor radii of the ellipse. (b/a) is called the Aspect Ratio of the ellipse.
- P = P (a, b) denotes the perimeter of the ellipse; Q = Q (a, b) is the arc-length of the Quarter Ellipse in the first quadrant. Obviously, P = 4Q.
- S (a, b) denotes an approximating formula for Q(a, b).
- Relative Error is: ((S (a, b)- Q (a, b)) / Q (a, b).
- Holder Norm of the pair (a, b) is N_p (a, b):$= (|a|^p + |b|^p)^{\frac{1}{p}}$,p> 0.
- Holder Mean of the pair (a, b) is H_p (a, b) $:= \left(\frac{|a|^p+|b|^p}{2}\right)^{\frac{1}{p}}$,$p > 0$.

II. HISTORY OF FORMULAE FOR P(A,B)

Down the annals of thehistory of Mathematics, one comes across a number of formulae for the perimeter of the ellipse. Some of them are 'simple but not exact', some others are 'exact but not so simple'. A number of such formulae can be found in [2], [4], [5]and [6]. Among these, the Formula of Takakazu Sekiis the oldest and,presumably, dates back to the 17th century. Particular attention is dawn to the formula of DavidW. Cantrell [6] for its similarity with my formula (6). The second formulae of S. Ramanujan (1914) [1] needs special attention for its accuracy, though it is not a formula of the type we are discussing in this paper.

III. ACTUAL FORMULA FOR P(A,B)

It is known from Integral Calculus that if (a cosθ, b sin θ), $0 \leq \theta < 2\pi$ is a parametric point on the ellipse.then

$$\mathbf{P}(a, b) = \int_0^{2\pi} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^{2\pi} \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}\ d\theta \qquad \ldots (1)$$

As the ellipse is symmetric about both the x- and y-axes, P (a, b) = 4 Q (a, b), where

$$\mathbf{Q}(a, b) = \int_0^{\pi/2} \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}\ d\theta \text{ ---- } (2)$$

The following properties of P (a, b) and Q (a, b) are obvious.

1. P (a, b) and Q (a, b) both are symmetric w.r.t. a and b
2. P (a, 0) = 4a; P (a, a) = $2\pi a$, and, as b/a increases from 0 to 1, P (a,b) increases from 4a to $2\pi a$.
3. (a) $[\frac{\partial Q}{\partial a}]\,(a, 0) = 1$;
   3(b) $[\frac{\partial Q}{\partial a}]\,(a, a) = \pi/4$.
4. (a) $[\frac{\partial Q}{\partial b}]\,(a, 0) = 0$;
   4(b) $[\frac{\partial Q}{\partial b}]\,(a, a) = \pi/4$

I shall focus on Q (a, b) for the remaining part of this paper. If needed, the equationP (a, b) = 4 Q (a, b) may be invoked.

The Partial Differential Equation (PDE) satisfied by Q (a, b).

It is easily seen that, Q (a, b) = $\int_0^{\pi/2}\sqrt{a^2sin^2\theta + b^2cos^2\theta}\ d\theta$satisfies the Partial Differential Equation:

$$a\frac{\partial Q}{\partial a} + b\frac{\partial Q}{\partial b} = Q \quad \ldots \quad (3)$$

This first order PDE is known as Lagrange's equation, and, Q is solved from the Auxiliary Equations: $\frac{da}{a} = \frac{db}{b} = \frac{dQ}{Q}$… (4)

Equation (4) has several solutions. For example, Q (a, b) = a, Q (a, b) = b, Q (a, b) = $\sqrt{ab}$,----- (4)

Q (a, b) = $(a^p + b^p)^{\frac{1}{p}}$, p $\neq$ 0are all solutions of (4) and each has physical dimension one. From among these, we choose two solutions, which are symmetric and linearly independent, namely, Q =$\sqrt{ab}$and Q= $(a^p + b^p)^{\frac{1}{p}}$,p ≠ 0.

Let U (a, b) $:= \sqrt{ab}$ and

N_p (a, b)$:= (a^p + b^p)^{\frac{1}{p}}$,p ≠ 0 … (5)

Then, the general solution of equation (3) is of the form Q(a, b) = F (U, N_p), where F is an arbitrary function.

IV. CONJECTURE

TheQuarter Perimeter Q (a, b) of the ellipseis likely to be a specific function of U (a, b) and N_p(a, b)for one or more 'p', satisfying i. a. the properties mentioned above.

The **conjecture above** will be verified now for a few well-known, simple, closed curves, which are also symmetric about the x- and y- axes. It may be noted that the partial differential equationgiven in equation (3) is satisfied by the integral form of theQuarter Perimeterin these cases.Further, eachcurve has explicitly known and distinct formula for Q (a, b),as given below.

V. ILLUSTRATIONS

1. Consider the closed curve C_1: |x/a| + |y/b| = 1. Its quarter perimeter (arc length in the first quadrant)is,obviously, $Q_{C_1}(a,b) = (a^2 + b^2)^{\frac{1}{2}}$ = N_2 (a, b).
2. Consider the closed curve C_2 formed by: |x| = a and |y| = b. Its quarter perimeter is$Q_{C_2}(a,b) = a + b$ = N_1 (a, b).
3. Next, consider theCurveC_3: (x/a) $^{(2/3)}$ + (y/b)$^{(2/3)}$ = 1.
   By integration, we get its quarter perimeter as
   $Q_{C_3}(a,b)$= (a+b) –ab/(a+b) = N_1(a, b) – U^2(a, b) / N_1(a, b)

Thus, for all the three curves above, Q (a, b) of the ellipse is a specific function of U (a, b) and N_p(a, b) for one or more 'p'.

Now we shall see some well-known Approximate FormulaeS (a, b)for the quarter perimeter Q (a, b) of the ellipse involving U (a, b) and N_p (a, b),for one or more p.

The YNOT Formula [3]

$$\mathbf{S\ (a, b) = (a^p + b^p)^{\frac{1}{p}} = N_p\ (a, b)}$$

Obviously, S (a, 0) = $a$andS (a, a) =$\frac{\pi a}{2}$, if p = ln(2)/ln ($\frac{\pi}{2}$). The formula is exact for the aspect ratios 0 and 1, but the Maximum Relative Error with this p is 0.00362,for all aspect ratios.

1. **Cantrell Formula ([5], [6])**
   S (a, b) = (a + b) – (2 -$\frac{\pi}{2}$) ab / H_p, where, H_p $(a, b) := \left(\frac{|a|^p + |b|^p}{2}\right)^{\frac{1}{p}}$,p ≠ 0 , where H_p (a, b) is the Holder Mean.

Here also**,** S (a, 0) = a and S (a, a) =$\frac{\pi a}{2}$. Thus the formula is exact for (b/a) = 0 and 1.

As H_p = $2^{-(\frac{1}{p})}N_p$, Cantrell Formula may be re-written as

S (a, b) = (a + b) – $2^{(\frac{1}{p})}$ (2 -$\frac{\pi}{2}$) ab/ N_p, which is also of the form N_1 – k. U^2/ N_p,

where k= $2^{(\frac{1}{p})}(2-\frac{\pi}{2})$. The relative error is minimized by optimizing the value of p.

The maximum Relative Error due to Cantrell Formula isless than **83 ppm**(i.e. 0.000083),bytakingp =0.82506([5], [6]).

ANew Approximation Formula for Q (a, b) (Discovered by the Author)

The quarter perimeter Q (a, b)of the ellipse: $(x/a)^2 + (y/b)^2 = 1$, $a \geq b \geq 0$ may be approximated by:

$$\boldsymbol{S(a,b) := (a^p + b^p)^{\frac{1}{p}} + \mathbf{k.\,(ab)^2/(a+b)^3}\ldots}$$

where k = 0.48251, and p = ln (2)/ ln($(\frac{\pi}{2}) - (0.48251/2^3)$) .

Obviously, using our notations, **$S(a, b) = N_p + k.U^4 / N_1^3$,** where k and p are as given above.Thus,thisS (a, b)also is a function of U (a, b), N_1 (a, b) and N_p (a, b). Further,S (a, 0) = a,and,S (a, a) = $\frac{\pi a}{2}$,for p = ln (2)/ ln $((\frac{\pi}{2}) - (\frac{0.48251}{2^3}))$.

The maximum absolute relative error, as per my formula, is less than 60 ppm, (see the attached table), which is an improvement upon Cantrell's formula, where the maximum absolute relative error is bounded by 83 ppm. The value, k =0.48251, is arrived at by minimizing the gap between the positive and negative relative errors. Note that the same k =0.48251works for all aspect ratios from 0 to 1.

Moreover, the properties 3 (a), 3 (b), 4 (a) and 4(b),which Q (a, b) is to possess,are possessed by both U (a, b) and N_p (a, b) and, hence, by S (a, b).

VI. DISCUSSION

An approximation formula for the ellipse perimeter,banking on partial differential equations is **not seen** in any related literature on the subject. Calculation of S(a, b) given in formula (6)is easy using a scientific calculator.

By changing the value of k, we can further reduce the absolute relative error in a certain range of (b/a).

For example, it can be verified that for k=0.49, the relative error is in the range ($-89*10^{-7}$, $62*10^{-7}$), for, $0.47 \leq \left(\frac{b}{a}\right) \leq 1$.

The attached Table givesS (a, b) for *a* = 100 and for all integral values of *b*from **(6)** 0 to 100. As no exact formula for Q (a, b) is available, we replaced Q (a, b) byS (a, b) got by Simpson's one-third rule method of Numerical Integration, with 250 equal sub-intervals of [0, $\frac{\pi}{2}$]. As the absolute relative errorin Simpson's one-third rule is of the order of (h^4), (where $h = \frac{\pi}{500}$) or $1.55855*10^{-09}$,our upper bound calculation for the relative error is vindicated.

Relative Error for other values of *a*and*b* may be got by suitable interpolation.

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Table1: New Formula Valuesfor S (a, b) and the Relative Error

a	b	k = LN(2)/LN((PI()/2)-(0.48251/8))	Q(a, b) by my Formula	Q(a, b) by Simpson's one-third Rule	Relative Error	Remarks
100	100	1.680645314	157.079632679490000	157.079632679490000	0.000000000000000	
100	99	1.680645314	156.295223019658000	156.295221198759000	0.000000011650382	
100	98	1.680645314	155.512810351330000	155.512803035381000	0.000000047044026	
100	97	1.680645314	154.732425130401000	154.732408602913000	0.000000106813360	
100	96	1.680645314	153.954098466208000	153.954068977126000	0.000000191544673	
100	95	1.680645314	153.177862139954000	153.177815915124000	0.000000301772348	
100	94	1.680645314	152.403748623820000	152.403681875157000	0.000000437972774	
100	93	1.680645314	151.631791100781000	151.631700037168000	0.000000600557885	
100	92	1.680645314	150.862023485152000	150.861904324107000	0.000000789868358	
100	91	1.680645314	150.094480443921000	150.094329424045000	0.000001006166432	
100	90	1.680645314	149.329197418888000	149.329010813121000	0.000001249628366	
100	89	1.680645314	148.566210649664000	148.565984779372000	0.000001520336514	
100	88	1.680645314	147.805557197556000	147.805288447480000	0.000001818271040	
100	87	1.680645314	147.047274970405000	147.046959804474000	0.000002143301241	
100	86	1.680645314	146.291402748409000	146.291037726448000	0.000002495176512	
100	85	1.680645314	145.537980210988000	145.537562006337000	0.000002873516946	
100	84	1.680645314	144.787047964740000	144.786573382793000	0.000003277803572	
100	83	1.680645314	144.038647572567000	144.038113570235000	0.000003707368271	
100	82	1.680645314	143.292821584000000	143.292225290119000	0.000004161383354	
100	81	1.680645314	142.549613566817000	142.548952303487000	0.000004638850860	
100	80	1.680645314	141.809068140012000	141.808339444872000	0.000005138591584	
100	79	1.680645314	141.071231008197000	141.070432657627000	0.000005659233861	
100	78	1.680645314	140.336148997506000	140.335279030736000	0.000006199202201	
100	77	1.680645314	139.603870093105000	139.602926837206000	0.000006756705755	
100	76	1.680645314	138.874443478382000	138.873425574122000	0.000007329726734	
100	75	1.680645314	138.147919575928000	138.146826004434000	0.000007916008822	
100	74	1.680645314	137.424350090413000	137.423180200603000	0.000008513045676	
100	73	1.680645314	136.703788053467000	136.702541590177000	0.000009118069609	
100	72	1.680645314	135.986287870683000	135.984965003428000	0.000009728040564	
100	71	1.680645314	135.271905370892000	135.270506723159000	0.000010339635501	
100	70	1.680645314	134.560697857822000	134.559224536798000	0.000010949238363	
100	69	1.680645314	133.852724164316000	133.851177790929000	0.000011552930744	
100	68	1.680645314	133.148044709260000	133.146427448379000	0.000012146483476	
100	67	1.680645314	132.446721557392000	132.445036148041000	0.000012725349324	
100	66	1.680645314	131.748818482188000	131.747068267573000	0.000013284657021	
100	65	1.680645314	131.054401032024000	131.052589989168000	0.000013819206906	
100	64	1.680645314	130.363536599826000	130.361669368569000	0.000014323468440	
100	63	1.680645314	129.676294496453000	129.674376407550000	0.000014791579929	

100	62	1.680645314	128.992746028061000	128.990783130063000	0.000015217350803	
100	61	1.680645314	128.312964577718000	128.310963662313000	0.000015594266836	
100	60	1.680645314	127.637025691583000	127.634994316991000	0.000015915498746	
100	59	1.680645314	126.965007169942000	126.962953681968000	0.000016173914633	
100	58	1.680645314	126.296989163481000	126.294922713731000	0.000016362096790	
100	57	1.680645314	125.633054275141000	125.630984835900000	0.000016472363440	
100	56	1.680645314	124.973287667991000	124.971226043167000	0.000016496796014	
100	55	1.680645314	124.317777179534000	124.315735011060000	0.000016427272659	
100	54	1.680645314	123.666613442960000	123.664603211928000	0.000016255508685	
100	53	1.680645314	123.019890015830000	123.017925037629000	0.000015973104737	
100	52	1.680645314	122.377703516801000	122.375797929393000	0.000015571603541	
100	51	1.680645314	121.740153770978000	121.738322515427000	0.000015042556149	
100	50	1.680645314	121.107343964590000	121.105602756846000	0.000014377598598	
100	49	1.680645314	120.479380809718000	120.477746102595000	0.000013568540050	
100	48	1.680645314	119.856374719889000	119.854863654071000	0.000012607463488	
100	47	1.680645314	119.238439997406000	119.237070340246000	0.000011486840085	
100	46	1.680645314	118.625695033398000	118.624485104170000	0.000010199658417	
100	45	1.680645314	118.018262521628000	118.017231101804000	0.000008739569758	
100	44	1.680645314	117.416269687243000	117.415435914284000	0.000007101050661	
100	43	1.680645314	116.819848531726000	116.819231774770000	0.000005279584071	
100	42	1.680645314	116.229136095476000	116.228755811240000	0.000003271860169	
100	41	1.680645314	115.644274739564000	115.644150306677000	0.000001075998109	
100	40	1.680645314	115.065412448371000	115.065562978324000	-0.000001308210287	
100	39	1.680645314	114.492703155034000	114.493147277843000	-0.000003879033988	
100	38	1.680645314	113.926307091788000	113.927062714469000	-0.000006632512617	
100	37	1.680645314	113.366391167567000	113.367475203505000	-0.000009562142373	
100	36	1.680645314	112.813129375461000	112.814557442816000	-0.000012658537933	
100	35	1.680645314	112.266703232978000	112.268489320333000	-0.000015909070888	
100	34	1.680645314	111.727302258356000	111.729458356001000	-0.000019297485884	
100	33	1.680645314	111.195124486628000	111.197660182088000	-0.000022803496543	
100	32	1.680645314	110.670377029589000	110.673299066356000	-0.000026402364364	
100	31	1.680645314	110.153276684345000	110.156588483269000	-0.000030064465230	
100	30	1.680645314	109.644050595812000	109.647751739223000	-0.000033754849977	
100	29	1.680645314	109.142936979241000	109.147022658760000	-0.000037432807776	
100	28	1.680645314	108.650185909754000	108.654646339879000	-0.000041051443956	
100	27	1.680645314	108.166060186960000	108.170879987965000	-0.000044557287566	
100	26	1.680645314	107.690836283981000	107.695993839586000	-0.000047889948554	
100	25	1.680645314	107.224805391785000	107.230272189460000	-0.000050981850218	
100	24	1.680645314	106.768274571615000	106.774014536549000	-0.000053758069875	
100	23	1.680645314	106.321568030676000	106.327536868368000	-0.000056136329948	
100	22	1.680645314	105.885028539140000	105.891173106713000	-0.000058027193321	
100	21	1.680645314	105.459019010251000	105.465276743059000	-0.000059334531721	

100	20	1.680645314	105.043924270011000	105.050222698445000	-0.000059956354892	Minimum
100	19	1.680645314	104.640153049025000	104.646409451062000	-0.000059786112778	
100	18	1.680645314	104.248140237061000	104.254261485833000	-0.000058714614493	
100	17	1.680645314	103.868349451519000	103.874232134835000	-0.000056632749002	
100	16	1.680645314	103.501275985319000	103.506806897050000	-0.000053435246402	
100	15	1.680645314	103.147450219514000	103.152507352688000	-0.000049025790104	
100	14	1.680645314	102.807441613646000	102.811895824480000	-0.000043323885801	
100	13	1.680645314	102.481863426635000	102.485580990886000	-0.000036274022311	
100	12	1.680645314	102.171378379473000	102.174224732294000	-0.000027857836245	
100	11	1.680645314	101.876705559280000	101.878550604060000	-0.000018110237815	
100	10	1.680645314	101.598629001857000	101.599354502522000	-0.000007140799951	
100	9	1.680645314	101.338008612030000	101.337518361821000	0.000004837795684	
100	8	1.680645314	101.095794455115000	101.094028165077000	0.000017471754470	
100	7	1.680645314	100.873046114651000	100.869998319400000	0.000030215081811	
100	6	1.680645314	100.670960055284000	100.666705836685000	0.000042260433220	
100	5	1.680645314	100.490910448959000	100.485640478649000	0.000052445008912	
100	4	1.680645314	100.334514543048000	100.328582826687000	0.000059122895919	
100	3	1.680645314	100.203747935717000	100.197736240671000	0.000059998312052	Maximum
100	2	1.680645314	100.101179024107000	100.095979045020000	0.000051949929827	
100	1	1.680645314	100.030576949544000	100.027463597804000	0.000031124969362	
100	0	1.680645314	100.000000000000000	100.000000000000000	0.000000000000000	
a	b	k = LN(2)/LN((PI()/2)-(0.48251/8))	Q(a, b) by my Formula	Q(a, b) by Simpson's one-third Rule	Relative Error	Remarks